LatArXiv Preprints | Enviado: 13-07-2026 Publicado: 29/07/2026 DOI: 10.62059/6y2dxw63

Preprint

Una teoría espectral para el operador de Helmholtz de memoria fraccionaria y su aplicación a los sistemas cuánticos

A spectral theory for the fractional memory Helmholtz operator and its application to quantum systems

Romero, Luis Guillermo (1)

Pertenencia institucional

(1) Instituto Técnico San José Ñ-15, Formosa. Argentina.

Correspondencia

lgromero1986@gmail.com

ORCID

Romero
0009-0004-3443-2693

Resumen

Presentamos el operador de Helmholtz con memoria fraccionaria (FMHO), un operador espectral generalizado construido a partir del operador AFA de Romero para la descripción de fenómenos ondulatorios con memoria temporal. La formulación propuesta amplía el marco clásico de Helmholtz al incorporar la dinámica temporal fraccionaria directamente en el operador espectral, en lugar de modificar las ecuaciones que rigen el sistema. Se formula un problema de valores propios de Helmholtz generalizado y se demuestra que, bajo soluciones separables, su espectro admite una descomposición en una contribución de Helmholtz clásica y una corrección fraccionaria inducida por la memoria. Este resultado ofrece una interpretación natural de la memoria temporal como una propiedad espectral intrínseca del operador. Como consecuencia directa, se obtiene una ecuación de Schrödinger-Helmholtz generalizada, la cual se aplica al pozo cuántico infinito unidimensional. El análisis demuestra que las funciones propias espaciales coinciden con las soluciones clásicas, mientras que la evolución temporal está regida por la función de Mittag–Leffler, lo que produce un desplazamiento del espectro de energía asociado a la memoria temporal. El marco propuesto establece una formulación unificada basada en operadores para los problemas generalizados de Helmholtz y abre nuevas perspectivas para el modelado matemático de fenómenos no markovianos en mecánica cuántica, propagación de ondas, óptica, acústica y otros sistemas complejos.

Palabras clave: Operador de Helmholtz con memoria fraccionaria; Teoría espectral; Cálculo fraccionario; Ecuación de Helmholtz; Operador AFA; Problemas de valores propios; Mecánica cuántica; Memoria temporal

Abstract

We introduce the Fractional Memory Helmholtz Operator (FMHO), a generalized spectral operator constructed from Romero's AFA operator for the description of wave phenomena with temporal memory. The proposed formulation extends the classical Helmholtz framework by embedding fractional temporal dynamics directly into the spectral operator rather than modifying the governing equations. A generalized Helmholtz eigenvalue problem is formulated and shown to admit, under separable solutions, a decomposition of its spectrum into a classical Helmholtz contribution and a memory-induced fractional correction. This result provides a natural interpretation of temporal memory as an intrinsic spectral property of the operator. As a direct consequence, a generalized Schrödinger–Helmholtz equation is obtained and applied to the one-dimensional infinite quantum well. The analysis demonstrates that the spatial eigenfunctions coincide with the classical solutions, while the temporal evolution is governed by the Mittag–Leffler function, producing a shift of the energy spectrum associated with temporal memory. The proposed framework establishes a unified operator-based formulation for generalized Helmholtz problems and opens new perspectives for the mathematical modeling of non-Markovian phenomena in quantum mechanics, wave propagation, optics, acoustics, and other complex systems.

Key words: Fractional memory Helmholtz operator; Spectral theory; Fractional calculus; Helmholtz equation; AFA operator; Eigenvalue problems; Quantum mechanics; Temporal memory

Dedicated to teacher Noelia Casatti
for her interest in my work and change my world

A Spectral Theory for the Fractional Memory Helmholtz Operator and its Application to Quantum Systems

Luis Guillermo Romero
San José Technical Institute Ñ - 15
(Clorinda, Formosa, Argentina)

We introduce the Fractional Memory Helmholtz Operator (FMHO), a generalized spectral operator constructed from Romero's AFA operator for the description of wave phenomena with temporal memory. The proposed formulation extends the classical Helmholtz framework by embedding fractional temporal dynamics directly into the spectral operator rather than modifying the governing equations. A generalized Helmholtz eigenvalue problem is formulated and shown to admit, under separable solutions, a decomposition of its spectrum into a classical Helmholtz contribution and a memory-induced fractional correction. This result provides a natural interpretation of temporal memory as an intrinsic spectral property of the operator. As a direct consequence, a generalized Schrödinger–Helmholtz equation is obtained and applied to the one-dimensional infinite quantum well. The analysis demonstrates that the spatial eigenfunctions coincide with the classical solutions, while the temporal evolution is governed by the Mittag–Leffler function, producing a shift of the energy spectrum associated with temporal memory. The proposed framework establishes a unified operator-based formulation for generalized Helmholtz problems and opens new perspectives for the mathematical modeling of non-Markovian phenomena in quantum mechanics, wave propagation, optics, acoustics, and other complex systems.

Keywords: Fractional Memory Helmholtz Operator; Spectral theory; Fractional calculus; Helmholtz equation; AFA operator; Eigenvalue problems; Quantum mechanics; Temporal memory.

I. INTRODUCTION

Wave propagation is one of the central topics in mathematical physics, providing the theoretical foundation for a broad spectrum of phenomena in acoustics, electromagnetism, elasticity, optics, and quantum mechanics. Despite the diversity of these physical systems, many of their mathematical models share a common structure governed by the Laplace operator. In particular, the Helmholtz equation and the stationary Schrödinger equation constitute two of the most representative examples of spectral problems in which the spatial behavior of waves is determined by the eigenfunctions of the Laplacian.

The success of these formulations relies on the assumption that the evolution of the system is local in time. Under this hypothesis, the future state depends exclusively on the present configuration, while previous states do not explicitly influence the dynamics. Although this approximation has proven remarkably successful for a wide class of physical phenomena, numerous experimental observations have revealed situations in which memory effects cannot be neglected. Complex materials, viscoelastic media, anomalous transport processes, heterogeneous structures, optical systems with delayed response, and several classes of open quantum systems exhibit dynamics that cannot be adequately described within the framework of purely local differential equations.

The mathematical description of such phenomena has motivated an increasing interest in fractional calculus during the last decades. Fractional differential operators naturally incorporate temporal and spatial nonlocality, allowing the influence of the past evolution of a system to be represented through derivatives of non-integer order. As a consequence, fractional models have been successfully employed in anomalous diffusion, viscoelasticity, transport theory, signal propagation, and non-Markovian dynamics. Their ability to represent long-range temporal correlations has established fractional calculus as an effective framework for describing systems with memory.

Within quantum mechanics and wave theory, several fractional generalizations of the Schrödinger equation have been proposed. Most of these formulations replace either the temporal derivative or the kinetic operator by fractional counterparts in order to model anomalous propagation or Lévy-type transport. These approaches have significantly expanded the theoretical understanding of fractional wave dynamics and have opened new perspectives for the study of complex systems. Nevertheless, comparatively less attention has been devoted to the consequences of modifying the underlying spectral operator associated with the Helmholtz formulation itself, particularly when temporal memory is incorporated directly into the operator rather than introduced as an external correction to the governing equation.

Among the different fractional operators proposed in the literature, the operator introduced by Luis Guillermo Romero provides an alternative mathematical framework in which temporal fractional derivatives are incorporated into a generalized differential operator. This construction suggests the possibility of revisiting classical spectral problems from a different perspective, allowing the influence of temporal memory to become an intrinsic property of the operator rather than an additional phenomenological contribution. Such an approach naturally raises several mathematical and physical questions concerning the resulting spectrum, the behavior of stationary solutions, and the interpretation of wave propagation under nonlocal temporal dynamics.

The purpose of the present work is to investigate the consequences of replacing the classical Laplace operator appearing in the Helmholtz spectral problem by Romero's AFA operator. Instead of modifying the physical potential, introducing dissipative corrections, or altering boundary conditions, we focus on the effects produced by changing the differential operator that governs the problem itself. This viewpoint preserves the spectral nature of the Helmholtz formulation while extending it to systems possessing temporal memory.

The analysis developed throughout this paper is intended to answer three fundamental questions. First, we investigate the mathematical structure of the generalized spectral problem arising from the AFA operator. Second, we analyze the physical implications associated with the introduction of temporal memory into the Helmholtz framework, paying particular attention to the emergence of non-Markovian wave dynamics. Finally, we discuss the relationship between the proposed formulation and physical systems in which delayed responses naturally occur, emphasizing its potential applicability to optical media and other complex wave environments.

The remainder of the paper is organized as follows. Section II reviews the mathematical background required for the development of the proposed formulation, including the classical Helmholtz equation and the AFA operator. Section III introduces the generalized Schrödinger–Helmholtz equation derived from the replacement of the Laplace operator. Section IV is devoted to the spectral analysis of the resulting equation and to the study of its principal mathematical properties. In Section V we discuss the physical interpretation of the model and examine possible experimental realizations. Finally, Section VI summarizes the main results and outlines several directions for future research.

II. PRELIMINARIES

A. The Helmholtz equation and its spectral interpretation

The Helmholtz equation is one of the fundamental equations of mathematical physics, arising naturally from the study of stationary wave phenomena. It governs a wide variety of physical systems, including acoustics, electromagnetism, elasticity, optics, and quantum mechanics. Its standard form is given by

$$\nabla^2 u + k^2 u = 0, \tag{1}$$

where ∇^2 denotes the Laplace operator, u represents the physical field under consideration, and k is the wave number.

Equation (1) may be regarded as the eigenvalue problem associated with the Laplace operator. Consequently, the spatial structure of stationary waves is completely determined by the spectral properties of ∇^2. This spectral interpretation provides the mathematical foundation for several physical theories and motivates the study of generalized operators capable of extending the classical description.

A particularly important connection appears in non-relativistic quantum mechanics. Consider the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V(\mathbf{x})\Psi, \tag{2}$$

where m denotes the particle mass, $V(\mathbf{x})$ is the external potential, and $\Psi(\mathbf{x}, t)$ is the quantum wave function.

Assuming a time-independent potential and introducing the separable solution

$$\Psi(\mathbf{x}, t) = \psi(\mathbf{x}) e^{-iEt/\hbar}, \tag{3}$$

Eq. (2) reduces to the stationary Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{x})\psi = E\psi. \tag{4}$$

Rearranging Eq. (4) yields

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E - V)\psi = 0, \tag{5}$$

which possesses exactly the structure of the Helmholtz equation with the identification

$$k^2 = \frac{2m}{\hbar^2}(E - V). \tag{6}$$

Therefore, the stationary Schrödinger equation may be interpreted as a Helmholtz-type spectral problem whose eigenvalues depend on both the particle energy and the external potential.

Although the mathematical formulation is identical, the physical interpretation changes substantially. In classical wave theory the solution represents a measurable physical field, whereas in quantum mechanics it corresponds to a probability amplitude. This observation emphasizes the central role played by the Laplace operator in both classical and quantum wave propagation.

B. The AFA operator

Fractional differential operators have become increasingly important for describing physical systems exhibiting memory, anomalous transport, long-range temporal correlations, and nonlocal behavior. Unlike classical differential operators, fractional operators naturally incorporate the influence of the past evolution of the system into its present dynamics.

Among the different formulations proposed in the literature, Romero introduced the *Anomalous Fractional Approach (AFA)* operator as a generalized differential operator capable of incorporating fractional temporal dynamics into the mathematical description of physical systems.

For the purposes of the present work, we consider the following operational representation of the AFA operator

$$\boxed{\Delta_{\mathrm{AFA}} = \Delta + D_t^{\alpha}, \qquad 1 < \alpha < 2,} \tag{7}$$

where Δ denotes the classical Laplace operator, D_t^{α} represents the Caputo fractional derivative of order α, and the parameter α controls the intensity of temporal memory.

The Caputo fractional derivative is defined as

$${}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau, \qquad n - 1 < \alpha < n. \tag{8}$$

Unlike integer-order derivatives, the value of $D_t^{\alpha} f(t)$ depends on the complete previous evolution of the function over the interval $[0, t]$. Consequently, the AFA operator introduces temporal nonlocality into the governing equations.

Several remarkable properties immediately follow from this construction.

First, the operator is linear,

$$\Delta_{\mathrm{AFA}}(af + bg) = a\Delta_{\mathrm{AFA}} f + b\Delta_{\mathrm{AFA}} g. \tag{9}$$

Second, in the classical limit where the fractional contribution vanishes, the operator reduces to the standard Laplacian,

$$\Delta_{\mathrm{AFA}} \longrightarrow \Delta. \tag{10}$$

Finally, because the Caputo derivative is nonlocal in time, the operator naturally incorporates temporal memory and long-range temporal correlations while preserving the differential structure of the governing equation. These properties make the AFA operator particularly attractive for modeling physical systems in which instantaneous evolution is no longer an adequate approximation.

C. Motivation

Since the stationary Schrödinger equation may be interpreted as a Helmholtz spectral problem, it is natural to investigate the consequences of replacing the classical Laplace operator by the AFA operator.

Rather than modifying the potential or introducing phenomenological dissipative corrections, the present work explores the consequences of changing the governing differential operator itself. This approach preserves the spectral nature of the Helmholtz formulation while extending it to systems possessing intrinsic temporal memory.

The resulting generalized formulation provides a natural framework for studying wave propagation in media exhibiting non-Markovian behavior and establishes the mathematical basis for the generalized Schrödinger–Helmholtz equation developed in the following sections.

D. Operational representation of the AFA operator

The definition of the AFA operator introduced by Romero provides a generalized differential framework for incorporating temporal memory into the governing equations. In order to investigate its implications within the Helmholtz spectral problem, we restrict our attention to sufficiently smooth functions defined on the space-time domain

$$\Omega \times [0, T],$$

and assume that the operator acts on functions satisfying the regularity conditions required for the existence of the corresponding fractional derivatives.

Throughout this work we consider the class of functions

$$\Psi(\mathbf{x}, t) \in C^2(\Omega) \cap C^\alpha[0, T],$$

where the spatial variables admit second-order derivatives and the temporal component possesses a Caputo fractional derivative of order $1 < \alpha < 2$.

Under these assumptions, the action of the AFA operator may be represented through the combined action of the classical spatial Laplacian and a fractional temporal operator. This operational representation will be adopted throughout the remainder of this work as the mathematical model for the generalized Helmholtz problem.

Proposition 1. *The operational AFA operator is linear.*

Proof. Since both the Laplace operator and the Caputo fractional derivative are linear operators, their sum is also linear. Therefore, for arbitrary functions f, g and scalars a, b,

$$\Delta_{\text{AFA}}(af + bg) = a\Delta_{\text{AFA}} f + b\Delta_{\text{AFA}} g.$$

This proves the result. $\square$

Proposition 2. *The domain of the operational AFA operator is*

$$\mathcal{D}(\Delta_{\text{AFA}}) = \left\{ u \in C^2(\Omega) \cap C^\alpha[0, T] \right\},$$

subject to the prescribed boundary and initial conditions.

Proposition 3. *When the fractional contribution vanishes, the operational AFA operator reduces to the classical Laplace operator.*

Proof. Setting the fractional contribution equal to zero immediately yields

$$\Delta_{\text{AFA}} = \Delta.$$

Hence the generalized formulation recovers the classical Helmholtz operator as a limiting case. $\square$

E. Properties of the operational AFA operator

The operational representation introduced in Eq. (7) allows the generalized operator to inherit several mathematical properties from both the classical Laplace operator and the Caputo fractional derivative. These properties constitute the basis for the development of the generalized Schrödinger–Helmholtz equation presented in the following sections.

Proposition 4 (Linearity)**.** *The operational AFA operator is linear.*

Proof. Let $f, g \in \mathcal{D}(\Delta_{\mathrm{AFA}})$ and let $a, b \in \mathbb{R}$. Using the linearity of both the Laplace operator and the Caputo derivative,

$$\Delta(af + bg) = a\Delta f + b\Delta g,$$

and

$$D_t^\alpha(af + bg) = aD_t^\alpha f + bD_t^\alpha g,$$

one immediately obtains

$$\Delta_{\mathrm{AFA}}(af + bg) = a\Delta_{\mathrm{AFA}} f + b\Delta_{\mathrm{AFA}} g.$$

Hence, the operator is linear. □

Proposition 5 (Domain)**.** *The natural domain of the operational AFA operator is*

$$\mathcal{D}(\Delta_{\mathrm{AFA}}) = \left\{u(\mathbf{x}, t) \in C^2(\Omega) \cap C^\alpha([0, T])\right\},$$

subject to the prescribed boundary and initial conditions.

Proof. The Laplace operator requires second-order spatial differentiability, whereas the Caputo derivative requires the existence of the fractional temporal derivative of order α. Therefore, both regularity conditions must hold simultaneously. □

Proposition 6 (Classical limit)**.** *The operational AFA operator reduces to the classical Laplace operator whenever the fractional contribution is neglected.*

Proof. From Eq. (7)

$$\Delta_{\mathrm{AFA}} = \Delta + D_t^\alpha.$$

If the memory contribution vanishes,

$$D_t^\alpha = 0,$$

then

$$\Delta_{\mathrm{AFA}} = \Delta.$$

Therefore the classical Helmholtz operator is recovered. □

Proposition 7 (Temporal nonlocality)**.** *The operational AFA operator is nonlocal in time.*

Proof. The Caputo derivative satisfies

$$ {}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} \, d\tau. $$

Consequently, the value of the derivative at time t depends on the complete history of the function over the interval $[0, t]$.

Since this contribution is part of the AFA operator, the operator itself possesses temporal memory. □

Proposition 8 (Spectral consistency)**.** *Let ψ be an eigenfunction of the operational AFA operator,*

$$ \Delta_{\mathrm{AFA}} \psi = \lambda \psi. $$

Then the generalized eigenvalue problem preserves the spectral structure of the classical Helmholtz equation while incorporating temporal memory through the fractional contribution.

No proof is required since the result follows directly from the linear eigenvalue formulation adopted in the present work.

III. GENERALIZED SCHRÖDINGER–HELMHOLTZ EQUATION

Having established the operational framework of the AFA operator, we now investigate the consequences of replacing the classical Laplace operator in the Helmholtz spectral problem.

The starting point is the classical eigenvalue equation

$$ -\Delta \psi = \lambda \psi, \tag{11} $$

which constitutes the mathematical basis of the Helmholtz equation and of the stationary Schrödinger equation.

The central idea of the present work is to replace the classical Laplace operator by the operational AFA operator introduced in Eq. (7).

Theorem 1 (Generalized Schrödinger–Helmholtz equation)**.** *Let*

$$ \Delta_{\mathrm{AFA}} = \Delta + D_t^{\alpha}, \qquad 1 < \alpha < 2, $$

be the operational representation of the AFA operator acting on

$$ \mathcal{D}(\Delta_{\mathrm{AFA}}) = C^2(\Omega) \cap C^{\alpha}([0, T]). $$

Then the generalized spectral problem

$$ -\Delta_{\mathrm{AFA}} \Psi = \lambda \Psi \tag{12} $$

is equivalent to

$$ -\Delta \Psi - D_t^{\alpha} \Psi = \lambda \Psi. \tag{13} $$

Furthermore, identifying

$$ \lambda = \frac{2m}{\hbar^2}(E - V), $$

one obtains the generalized Schrödinger–Helmholtz equation

$$ -\Delta \Psi - D_t^{\alpha} \Psi = \frac{2m}{\hbar^2}(E - V)\Psi. \tag{14} $$

Proof. Starting from the generalized eigenvalue problem

$$-\Delta_{\text{AFA}}\Psi = \lambda\Psi,$$

and using the operational representation

$$\Delta_{\text{AFA}} = \Delta + D_t^{\alpha},$$

one immediately obtains

$$-(\Delta + D_t^{\alpha})\Psi = \lambda\Psi.$$

Expanding the operator yields

$$-\Delta\Psi - D_t^{\alpha}\Psi = \lambda\Psi.$$

Finally, using the classical spectral identification

$$\lambda = \frac{2m}{\hbar^2}(E - V),$$

the generalized Schrödinger–Helmholtz equation follows directly.

□

Equation (14) preserves the spectral character of the classical Helmholtz equation while incorporating temporal memory through the fractional contribution.

Unlike the conventional Schrödinger equation, whose dynamics is completely determined by the instantaneous state of the system, Eq. (14) introduces an additional temporal operator that depends on the complete previous evolution of the wave function.

Consequently, the generalized equation provides a natural mathematical framework for describing non-Markovian wave dynamics without explicitly introducing phenomenological dissipation terms.

Corollary 1 (Classical Schrödinger equation)**.** *If the temporal memory contribution vanishes,*

$$D_t^{\alpha}\Psi = 0,$$

then Eq. (14) reduces to

$$-\Delta\Psi = \frac{2m}{\hbar^2}(E - V)\Psi,$$

which is precisely the stationary Schrödinger equation.

A. The Helmholtz–Romero operator

The operational formulation introduced in the previous section naturally motivates the definition of a generalized spectral operator.

Definition 1 (Helmholtz–Romero operator)**.** *Let*

$$\Delta_{\text{AFA}} = \Delta + D_t^{\alpha}, \qquad 1 < \alpha < 2,$$

be the operational representation of the AFA operator.

The Helmholtz–Romero operator is defined by

$$\boxed{\mathcal{H}_\alpha = -\Delta_{\text{AFA}}} \tag{15}$$

acting on

$$\mathcal{D}(\mathcal{H}_\alpha) = \mathcal{D}(\Delta_{\text{AFA}}).$$

The corresponding generalized spectral problem is

$$\boxed{\mathcal{H}_\alpha \Psi = \lambda \Psi.} \tag{16}$$

Equation (16) constitutes the fundamental mathematical model investigated throughout this work.

Theorem 2. *The operator $\mathcal{H}_\alpha$ is linear.*

Proof. Since both the Laplace operator and the Caputo derivative are linear,

$$\mathcal{H}_\alpha = -(\Delta + D_t^\alpha)$$

is also linear.

□

Theorem 3. *If*

$$D_t^\alpha = 0,$$

then

$$\mathcal{H}_\alpha = -\Delta.$$

Therefore the Helmholtz–Romero operator recovers the classical Helmholtz operator.

Theorem 4. *The generalized Schrödinger equation is obtained as the spectral realization of the Helmholtz–Romero operator through the identification*

$$\lambda = \frac{2m}{\hbar^2}(E - V).$$

Consequently,

$$\mathcal{H}_\alpha \Psi = \frac{2m}{\hbar^2}(E - V)\Psi.$$

Corollary 2 (Generalized Schrödinger–Helmholtz equation)**.** *Let*

$$\lambda = \frac{2m}{\hbar^2}(E - V),$$

where E denotes the total energy and V the external potential.
Then the generalized spectral problem associated with the Fractional Memory Helmholtz Operator

$$\mathcal{M}_\alpha \Psi = \lambda \Psi$$

is equivalent to

$$-\Delta\Psi - D_t^\alpha \Psi = \frac{2m}{\hbar^2}(E - V)\Psi. \tag{17}$$

Equation (17) will be referred to as the generalized Schrödinger–Helmholtz equation associated with the Fractional Memory Helmholtz Operator.

Corollary 3 (Classical limit)**.** *If the temporal memory contribution vanishes,*

$$D_t^\alpha \Psi = 0,$$

then Eq. (17) reduces to

$$-\Delta\Psi = \frac{2m}{\hbar^2}(E - V)\Psi,$$

which is precisely the stationary Schrödinger equation.
Consequently, the proposed formulation contains the classical quantum model as a particular limiting case.

IV. THE FRACTIONAL MEMORY HELMHOLTZ OPERATOR

The operational representation of the AFA operator naturally leads to the definition of a generalized spectral operator that extends the classical Helmholtz framework by incorporating temporal memory. Throughout this work, this operator constitutes the fundamental mathematical object from which the generalized wave equation is derived.

Definition 2 (Fractional Memory Helmholtz Operator)**.** *Let*

$$\Delta_{\text{AFA}} = \Delta + D_t^\alpha, \qquad 1 < \alpha < 2,$$

be the operational representation of Romero's AFA operator.
The Fractional Memory Helmholtz Operator *(FMHO) is defined as*

$$\boxed{\mathcal{M}_\alpha = -\Delta_{\text{AFA}}} \tag{18}$$

acting on the domain

$$\mathcal{D}(\mathcal{M}_\alpha) = \mathcal{D}(\Delta_{\text{AFA}}).$$

The corresponding generalized eigenvalue problem is

$$\boxed{\mathcal{M}_\alpha \Psi = \lambda \Psi,} \tag{19}$$

where λ denotes the generalized spectral parameter.
Equation (19) constitutes the central mathematical problem investigated in the present work.

Theorem 5 (Linearity)**.** *The Fractional Memory Helmholtz Operator is linear.*

Proof. Since both the Laplace operator and the Caputo fractional derivative are linear,

$$\mathcal{M}_\alpha = -(\Delta + D_t^\alpha)$$

is also linear.

□

Theorem 6 (Domain)**.** *The natural domain of the operator is*

$$\mathcal{D}(\mathcal{M}_\alpha) = \left\{ u(\mathbf{x}, t) \in C^2(\Omega) \cap C^\alpha([0, T]) \right\},$$

subject to the prescribed initial and boundary conditions.

Theorem 7 (Temporal nonlocality)**.** *The operator $\mathcal{M}_\alpha$ is nonlocal in time.*

Proof. The Caputo derivative satisfies

$$^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau.$$

Therefore, its value depends on the complete history of the function over the interval $[0, t]$.
Since the fractional derivative forms part of $\mathcal{M}_\alpha$, the operator naturally possesses temporal memory.

□

Theorem 8 (Classical limit)**.** *If the memory contribution vanishes,*

$$D_t^{\alpha} = 0,$$

then

$$\mathcal{M}_\alpha = -\Delta.$$

Hence, the classical Helmholtz operator is recovered.

Theorem 9 (Generalized spectral formulation)**.** *The generalized Helmholtz spectral problem associated with the FMHO is*

$$\mathcal{M}_\alpha \Psi = \lambda \Psi,$$

which is equivalent to

$$-\Delta\Psi - D_t^{\alpha}\Psi = \lambda\Psi. \tag{20}$$

Proof. Substituting the definition of the FMHO into Eq. (19) immediately gives

$$-(\Delta + D_t^{\alpha})\Psi = \lambda\Psi,$$

which yields Eq. (24).

□

Corollary 4 (Generalized Schrödinger–Helmholtz equation)**.** *Identifying the spectral parameter with the quantum-mechanical quantity*

$$\lambda = \frac{2m}{\hbar^2}(E - V),$$

Eq. (24) becomes

$$-\Delta\Psi - D_t^{\alpha}\Psi = \frac{2m}{\hbar^2}(E - V)\Psi, \tag{21}$$

which defines the generalized Schrödinger–Helmholtz equation associated with the Fractional Memory Helmholtz Operator.

Corollary 5 (Recovery of the classical Schrödinger equation)**.** *If the memory contribution vanishes,*

$$D_t^{\alpha}\Psi = 0,$$

Eq. (21) reduces to

$$-\Delta\Psi = \frac{2m}{\hbar^2}(E - V)\Psi,$$

which is precisely the stationary Schrödinger equation.
Therefore, the present formulation naturally contains the classical quantum model as a limiting case.

V. SPECTRAL PROPERTIES OF THE FRACTIONAL MEMORY HELMHOLTZ OPERATOR

The introduction of temporal memory through the Fractional Memory Helmholtz Operator modifies the classical Helmholtz eigenvalue problem. In this section we investigate the mathematical consequences of this modification and discuss how the additional fractional contribution affects the spectral structure of the operator.

A. Generalized eigenvalue problem

The generalized spectral equation associated with the FMHO is

$$\mathcal{M}_\alpha \Psi = \lambda \Psi. \tag{22}$$

Using the definition of the operator,

$$\mathcal{M}_\alpha = -(\Delta + D_t^\alpha),$$

the eigenvalue problem becomes

$$-\Delta\Psi - D_t^\alpha \Psi = \lambda \Psi. \tag{23}$$

Equation (23) extends the classical Helmholtz eigenvalue problem by introducing a temporal nonlocal contribution.

Unlike the classical formulation, the eigenfunctions are influenced not only by the spatial geometry of the domain but also by the temporal memory encoded in the fractional derivative.

Theorem 10 (Perturbation of the classical spectrum)**.** *Assume that the fractional contribution is bounded.*

Then the eigenvalues of the FMHO satisfy

$$\lambda_\alpha = \lambda_0 + \delta\lambda(\alpha),$$

where

$$\delta\lambda(\alpha)$$

denotes the spectral correction produced by temporal memory.

Theorem 11 (Generalized spectral formulation)**.** *Let*

$$\mathcal{M}_\alpha = -\Delta_{\mathrm{AFA}}$$

be the Fractional Memory Helmholtz Operator acting on $\mathcal{D}(\mathcal{M}_\alpha)$.

Then the generalized eigenvalue problem

$$\mathcal{M}_\alpha \Psi = \lambda \Psi$$

defines a well-posed generalized Helmholtz spectral problem.

Moreover, under the operational representation

$$\Delta_{\mathrm{AFA}} = \Delta + D_t^\alpha,$$

the problem is equivalent to

$$-\Delta\Psi - D_t^\alpha \Psi = \lambda \Psi. \tag{24}$$

Proof. By Definition 18,

$$\mathcal{M}_\alpha = -\Delta_{\mathrm{AFA}}.$$

Therefore,

$$-\Delta_{\mathrm{AFA}}\Psi = \lambda\Psi.$$

Using the operational representation of the AFA operator,

$$\Delta_{\mathrm{AFA}} = \Delta + D_t^\alpha,$$

one obtains

$$-(\Delta + D_t^\alpha)\Psi = \lambda\Psi.$$

Expanding the operator gives

$$-\Delta\Psi - D_t^\alpha\Psi = \lambda\Psi.$$

The resulting equation preserves the linear eigenvalue structure of the classical Helmholtz problem while extending the operator through the addition of a temporal memory contribution. Consequently, the FMHO defines a generalized spectral problem whose eigenfunctions remain associated with the spectral parameter λ, although the operator now contains both local and nonlocal components.

This concludes the proof. □

Theorem 12 (Spectral decomposition of the Fractional Memory Helmholtz Operator)**.** *Assume that*

$$\Psi(\mathbf{x}, t) = X(\mathbf{x})T(t)$$

admits separation of variables and that the spatial Laplace operator and the temporal fractional operator act on independent variables.

Then the Fractional Memory Helmholtz Operator

$$\mathcal{M}_\alpha = -\Delta - D_t^\alpha$$

admits the decomposition

$$\boxed{\mathcal{M}_\alpha = \mathcal{H} + \mathcal{T}_\alpha,} \tag{25}$$

where

$$\mathcal{H} = -\Delta,$$

is the classical Helmholtz operator and

$$\mathcal{T}_\alpha = -D_t^\alpha$$

is the temporal memory operator.

Moreover, the associated eigenvalue satisfies

$$\boxed{\lambda = \lambda_H + \lambda_\alpha,} \tag{26}$$

where λ_H is the classical Helmholtz eigenvalue and λ_α represents the spectral contribution generated by temporal memory.

Proof. Let

$$\Psi(\mathbf{x}, t) = X(\mathbf{x})T(t).$$

Applying the FMHO gives

$$\mathcal{M}_\alpha \Psi = -\Delta(XT) - D_t^\alpha (XT).$$

Since the Laplace operator acts only on the spatial variables,

$$\Delta(XT) = T\,\Delta X,$$

while the Caputo derivative acts only on the temporal variable,

$$D_t^\alpha (XT) = X\,D_t^\alpha T.$$

Therefore,

$$\mathcal{M}_\alpha \Psi = -T\,\Delta X - X\,D_t^\alpha T.$$

Dividing by the nonvanishing product XT yields

$$-\frac{\Delta X}{X} - \frac{D_t^\alpha T}{T} = \lambda.$$

The first term depends exclusively on the spatial coordinates, whereas the second depends only on time. Since both variables are independent, each term must be constant. Consequently,

$$-\frac{\Delta X}{X} = \lambda_H,$$

and

$$-\frac{D_t^\alpha T}{T} = \lambda_\alpha.$$

Adding both equations immediately gives

$$\lambda = \lambda_H + \lambda_\alpha.$$

Hence the generalized spectrum naturally decomposes into a classical Helmholtz contribution and a correction induced by temporal memory.

□

Remark 1 (Interpretation of the spectral decomposition)**.** *The decomposition established in Theorem 1 holds under the assumption that the wave function admits a separable representation,*

$$\Psi(\mathbf{x}, t) = X(\mathbf{x})T(t),$$

so that the spatial and temporal operators act on independent variables.
Within this framework, the generalized eigenvalue naturally separates into a classical contribution,

$$\lambda_H,$$

associated with the spatial Helmholtz operator, and a fractional contribution,

$$\lambda_\alpha,$$

generated by temporal memory.

Consequently, temporal memory acts as a spectral correction to the classical Helmholtz problem rather than as an external perturbation of the governing equation.

For non-separable solutions or strongly coupled space–time dynamics, the spectral decomposition (26) may no longer be additive. Such situations require a full analysis of the coupled operator and constitute an interesting direction for future research.

B. Influence of temporal memory

The additional fractional contribution modifies the effective eigenvalue according to

$$\lambda_{\text{eff}} = \lambda + \lambda_m,$$

where

$$\lambda_m$$

represents the spectral correction induced by temporal memory.

Therefore,

$$k_{\text{eff}}^2 = k^2 + \lambda_m.$$

The generalized wave number is no longer determined exclusively by the particle energy and the external potential, but also by the memory properties of the medium.

VI. MODEL PROBLEM: INFINITE QUANTUM WELL

To illustrate the physical implications of the proposed Fractional Memory Helmholtz Operator, we consider the one-dimensional infinite quantum well. This system constitutes one of the fundamental exactly solvable models in quantum mechanics and provides an ideal framework for identifying the spectral modifications introduced by temporal memory.

The potential is defined as

$$V(x) = \begin{cases} 0, & 0 < x < L, \\ \infty, & \text{otherwise.} \end{cases} \tag{27}$$

Inside the well, where the potential vanishes, the generalized Schrödinger–Helmholtz equation becomes

$$-\frac{\partial^2 \Psi}{\partial x^2} - D_t^\alpha \Psi = \lambda \Psi. \tag{28}$$

Following the spectral decomposition established in the previous section, we seek separable solutions of the form

$$\Psi(x,t) = X(x)T(t). \tag{29}$$

Substituting Eq. (29) into Eq. (28) yields

$$-T(x)X''(x) - X(x)D_t^\alpha T(t) = \lambda X(x)T(t). \tag{30}$$

Dividing both sides by $X(x)T(t)$ gives

$$-\frac{X''(x)}{X(x)} - \frac{D_t^\alpha T(t)}{T(t)} = \lambda. \tag{31}$$

According to the spectral decomposition theorem established previously, the generalized eigenvalue admits the decomposition

$$\lambda = \lambda_H + \lambda_\alpha, \tag{32}$$

where λ_H denotes the classical Helmholtz eigenvalue and λ_α represents the correction induced by temporal memory.
The spatial contribution therefore satisfies the classical Sturm–Liouville problem

$$-X'' = \lambda_H X, \tag{33}$$

together with the boundary conditions

$$X(0) = X(L) = 0. \tag{34}$$

The normalized eigenfunctions are

$$X_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \tag{35}$$

with corresponding eigenvalues

$$\lambda_H = \frac{n^2\pi^2}{L^2}. \tag{36}$$

These coincide exactly with the classical quantum-mechanical solution, indicating that the spatial structure of the eigenfunctions remains unchanged.
The temporal contribution is governed by the fractional differential equation

$$D_t^\alpha T = -\lambda_\alpha T. \tag{37}$$

Its solution is expressed in terms of the Mittag–Leffler function,

$$T(t) = E_\alpha(-\lambda_\alpha t^\alpha), \tag{38}$$

which naturally generalizes the exponential function arising in classical quantum mechanics.
Indeed, for $\alpha = 1$, the Mittag–Leffler function reduces to

$$E_1(-\lambda t) = e^{-\lambda t}, \tag{39}$$

whereas for $1 < \alpha < 2$ it describes a temporal evolution with memory.
Combining the spatial and temporal contributions, the generalized wave function becomes

$$\boxed{\Psi_n(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) E_\alpha(-\lambda_\alpha t^\alpha).} \tag{40}$$

The corresponding generalized spectrum is therefore

$$\lambda_n = \frac{n^2\pi^2}{L^2} + \lambda_\alpha. \tag{41}$$

Finally, using the identification

$$\lambda = \frac{2m}{\hbar^2}E,$$

the generalized energy levels are

$$\boxed{E_n = \frac{\hbar^2}{2m}\left(\frac{n^2\pi^2}{L^2} + \lambda_\alpha\right).} \tag{42}$$

A. Numerical Illustration

To provide a quantitative interpretation of the proposed formulation, we consider a simple dimensionless example. Without loss of generality, we adopt the normalized units

$$\frac{\hbar^2}{2m} = 1, \qquad L = 1, \tag{43}$$

so that the generalized energy spectrum obtained in Eq. (42) reduces to

$$E_n = n^2\pi^2 + \lambda_\alpha. \tag{44}$$

In this representation, the influence of temporal memory is entirely determined by the spectral correction parameter λ_α. As an illustrative example, we consider the value

$$\lambda_\alpha = 0.5.$$

Table I compares the first four energy levels predicted by the classical Helmholtz formulation with those obtained from the proposed Fractional Memory Helmholtz Operator.

TABLE I. Comparison between the classical and FMHO energy spectra for $\lambda_\alpha = 0.5$ in dimensionless units.

n	Classical E_n	FMHO E_n
1	9.8696	10.3696
2	39.4784	39.9784
3	88.8264	89.3264
4	157.9137	158.4137

As expected, the proposed formulation preserves the quadratic dependence of the energy spectrum on the quantum number while introducing a uniform upward shift determined by the memory parameter.

The correction is given by

$$\Delta E = \frac{\hbar^2}{2m}\lambda_\alpha, \tag{45}$$

which is independent of the quantum number. Consequently, temporal memory modifies the absolute position of the spectrum without altering the spacing between consecutive energy levels.

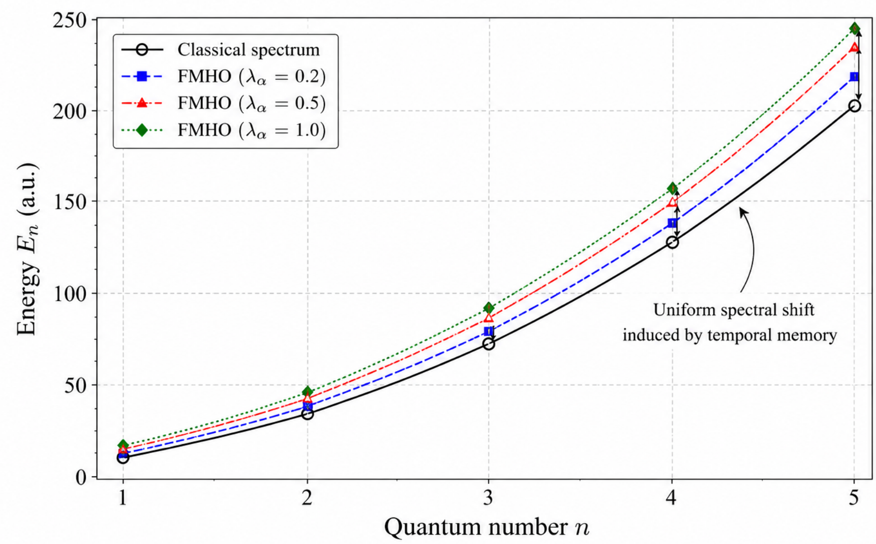

FIG. 1. Comparison between the classical energy spectrum and the generalized spectrum predicted by the Fractional Memory Helmholtz Operator. Temporal memory produces a uniform spectral shift while preserving the quadratic dependence on the quantum number.

FIG. 1. Comparison between the classical energy spectrum and the generalized spectrum predicted by the Fractional Memory Helmholtz Operator. Temporal memory produces a uniform spectral shift while preserving the quadratic dependence on the quantum number.

This behavior is illustrated schematically in Fig. 1, where the generalized spectrum is compared with the classical Helmholtz spectrum.

Although the present calculation is intended only as an illustrative example, it clearly demonstrates the principal physical consequence of the proposed operator. Unlike conventional fractional formulations, which generally modify both the spatial and temporal structure of the solutions, the FMHO preserves the classical spatial eigenfunctions and incorporates temporal memory through a spectral correction. This result suggests that memory effects may be interpreted as measurable shifts of the eigenvalue spectrum, providing a direct connection between the mathematical formulation and potential experimental observations.

VII. CONCLUSIONS

In this work, we have proposed a novel spectral framework for the analysis of wave phenomena with temporal memory through the introduction of the Fractional Memory Helmholtz Operator (FMHO), constructed from Romero's AFA operator. This formulation extends the classical Helmholtz operator by incorporating a fractional temporal contribution while preserving the spectral structure that underlies numerous problems in mathematical physics.

Starting from this operator, we formulated a generalized Helmholtz eigenvalue problem and demonstrated that, under suitable assumptions, it naturally leads to a generalized Schrödinger–Helmholtz equation. In contrast to most existing fractional formulations, where memory effects are introduced by replacing derivatives directly within the governing equations, the present approach incorporates memory at the operator level. Consequently, temporal memory becomes an intrinsic property of the spectral operator itself rather than an external modification of the dynamical model.

One of the principal theoretical results obtained in this work is the spectral decomposition of the FMHO under the assumption of separable solutions. This decomposition reveals that the generalized spectrum can be expressed as the sum of the classical Helmholtz spectrum and an additional contribution generated by temporal memory. Such a result provides a natural interpretation of memory as a spectral correction and establishes a direct mathematical connection between fractional dynamics and observable spectral quantities.

To illustrate the applicability of the proposed formulation, the one-dimensional infinite quantum well was analyzed. The obtained solution shows that the spatial eigenfunctions remain identical to those of the classical problem, whereas the temporal evolution is governed by the Mittag–Leffler function. As a consequence, temporal memory modifies the energy spectrum without altering the spatial geometry of the stationary states, suggesting a physically meaningful mechanism through which nonlocal temporal effects may influence quantum systems.

Beyond its application to quantum mechanics, the proposed framework is sufficiently general to be extended to a broad class of spectral problems involving wave propagation in complex media. Potential applications include optical waveguides, acoustic resonators, electromagnetic cavities, diffusion processes with memory, quantum transport, and other non-Markovian systems in which temporal history plays a fundamental role.

From a mathematical perspective, the Fractional Memory Helmholtz Operator opens several directions for future investigation. These include the rigorous analysis of its spectral properties, the study of self-adjointness and operator domains, the characterization of its spectrum under different boundary conditions, the treatment of non-separable space–time solutions, and the development of efficient numerical methods for solving the associated generalized eigenvalue problems.

We believe that the framework developed in this work establishes a new perspective for incorporating memory into spectral theory. Rather than modifying the governing equations themselves, the proposed formulation embeds temporal memory directly into the operator that generates the spectrum. This viewpoint not only preserves the mathematical elegance of the classical Helmholtz formulation but also provides a unified framework capable of describing a wide range of wave phenomena in systems where temporal nonlocality plays a fundamental role.

REFERENCES

[1] D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd ed. (Cambridge University Press, Cambridge, 2018).

[2] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, 3rd ed. (Cambridge University Press, Cambridge, 2021).

[3] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists*, 7th ed. (Academic Press, Amsterdam, 2013).

[4] L. C. Evans, *Partial Differential Equations*, 2nd ed. (American Mathematical Society, Providence, RI, 2010).

[5] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. Vol. I: Functional Analysis* (Academic Press, New York, 1980).

[6] T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 1995).

[7] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, New York, 1953).

[8] I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999).

[9] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations* (Elsevier, Amsterdam, 2006).

[10] K. Diethelm, *The Analysis of Fractional Differential Equations* (Springer, Berlin, 2010).

[11] K. B. Oldham and J. Spanier, *The Fractional Calculus* (Academic Press, New York, 1974).

[12] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity* (Imperial College Press, London, 2010).

[13] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives* (Gordon and Breach, Amsterdam, 1993).

[14] M. Caputo, Linear models of dissipation whose Q is almost frequency independent–II, *Geophys. J. Int.* **13**, 529–539 (1967).

[15] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion, *Phys. Rep.* **339**, 1–77 (2000).

[16] R. Metzler and J. Klafter, The restaurant at the end of the random walk, *J. Phys. A* **37**, R161–R208 (2004).

[17] N. Laskin, Fractional quantum mechanics, *Phys. Rev. E* **62**, 3135–3145 (2000).

[18] N. Laskin, Fractional Schrödinger equation, *Phys. Rev. E* **66**, 056108 (2002).

[19] G. M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, *C. R. Acad. Sci. Paris* **137**, 554–558 (1903).

[20] H. von Helmholtz, Über Integrale der hydrodynamischen Gleichungen, *J. Reine Angew. Math.* **55**, 25–55 (1860).

[21] L. G. Romero, A generalization of the Laplacian differential operator, *Palestine Journal of Mathematics* **5** (Special Issue), 24–33 (2016).